A New Approach to Assessing Model Risk in High Dimensions

Carole Bernard (University of Waterloo)
and Steven Vanduffel (Vrije Universiteit Brussels)

University of Waterloo

WATRISQ
Waterloo Research Institute in Insurance, Securities & Quantitative Finance

ARIA meeting 2014, Seattle

Objectives and Findings

- Model uncertainty on the risk assessment of the sum of d dependent risks (portfolio).
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?
- A non-parametric method based on the data at hand.
- Analytical expressions for the maximum and minimum

Objectives and Findings

- Model uncertainty on the risk assessment of the sum of d dependent risks (portfolio).
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?
- A non-parametric method based on the data at hand.
- Analytical expressions for the maximum and minimum
- Implications:
  - Current VaR based regulation is subject to high model risk, even if one knows the multivariate distribution almost completely.
  - We can identify for which risk measures it is meaningful to develop accurate multivariate models.

Model Risk

1. Goal: Assess the risk of a portfolio sum $S=\sum_{i=1}^{d} X_i$.
2. Choose a risk measure $\rho(\cdot)$: variance, Value-at-Risk...
3. "Fit" a multivariate distribution for $(X_1, X_2, ..., X_d)$ and compute $\rho(S)$
4. How about model risk? How wrong can we be?

$$\rho_{\mathcal{F}}^{+} := \sup\left\{\rho\left(\sum_{i=1}^{d} X_i\right)\right\}, \quad \rho_{\mathcal{F}}^{-} := \inf\left\{\rho\left(\sum_{i=1}^{d} X_i\right)\right\}$$

where the bounds are taken over all other (joint distributions of) random vectors $(X_1, X_2, ..., X_d)$ that "agree" with the available information $\mathcal{F}$

Assessing Model Risk on Dependence with d Risks

- Marginals known and dependence fully unknown
- A challenging problem in $d \geqslant 3$ dimensions
  - Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
  - Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds on VaR
- **Issues**
  - bounds are generally very wide
  - ignore all information on dependence.
- **Our answer:**
  - We incorporate in a natural way dependence information.

Rearrangement Algorithm

$N = 4$ observations of $d = 3$ variables: X_1, X_2, X_3

$$\mathbf{M} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{6} & \mathbf{3} \\ \mathbf{4} & \mathbf{0} & \mathbf{0} \\ \mathbf{6} & \mathbf{3} & \mathbf{4} \end{bmatrix}$$

Each column: **marginal** distribution

Interaction among columns: **dependence** among the risks

Same marginals, different dependence ⇒ Effect on the sum!

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 4 & 0 & 0 \\ 6 & 3 & 4 \end{bmatrix} \qquad \begin{matrix} X_1 + X_2 + X_3 \\ S_N = \begin{bmatrix} 4 \\ 9 \\ 4 \\ 13 \end{bmatrix} \end{matrix}$$

$$\begin{bmatrix} 6 & 6 & 4 \\ 4 & 3 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} X_1 + X_2 + X_3 \\ S_N = \begin{bmatrix} 16 \\ 10 \\ 3 \\ 0 \end{bmatrix} \end{matrix}$$

Aggregate Risk with Maximum Variance

comonotonic scenario

Rearrangement Algorithm: Sum with Minimum Variance

minimum variance with $d = 2$ risks X_1 and X_2

Antimonotonicity: $var(\mathbf{X_1^a} + X_2) \leqslant var(\mathbf{X_1} + X_2)$

How about in d dimensions?

Rearrangement Algorithm: Sum with Minimum Variance

minimum variance with $d = 2$ risks X_1 and X_2

Antimonotonicity: $var(\mathbf{X_1^a} + X_2) \leqslant var(\mathbf{X_1} + X_2)$

How about in d dimensions?
Use of the rearrangement algorithm on the original matrix M.

Aggregate Risk with Minimum Variance

- Columns of M are rearranged such that they become anti-monotonic with the sum of all other columns.

$$\forall k \in \{1, 2, ..., d\}, \mathbf{X_k^a} \textit{ antimonotonic with } \sum_{j \neq k} X_j$$

- After each step, $var\left(\mathbf{X_k^a} + \sum_{j \neq k} X_j\right) \leqslant var\left(\mathbf{X_k} + \sum_{j \neq k} X_j\right)$ where $\mathbf{X_k^a}$ is antimonotonic with $\sum_{j \neq k} X_j$

Aggregate risk with minimum variance
Step 1: First column

$$\begin{array}{c} \downarrow \\ \left[\begin{array}{ccc} 6 & 6 & 4 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \end{array} \quad \begin{array}{c} X_2 + X_3 \\ 10 \\ 5 \\ 2 \\ 0 \end{array} \quad \text{becomes} \quad \left[\begin{array}{ccc} 0 & 6 & 4 \\ 1 & 3 & 2 \\ 4 & 1 & 1 \\ 6 & 0 & 0 \end{array}\right]$$

Aggregate risk with minimum variance

$$\begin{bmatrix} \mathbf{6} & 6 & 4 \\ \mathbf{4} & 3 & 2 \\ \mathbf{1} & 1 & 1 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \quad \begin{matrix} X_2+X_3 \\ 10 \\ 5 \\ 2 \\ 0 \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} \mathbf{0} & 6 & 4 \\ \mathbf{1} & 3 & 2 \\ \mathbf{4} & 1 & 1 \\ \mathbf{6} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \mathbf{6} & 4 \\ 1 & \mathbf{3} & 2 \\ 4 & \mathbf{1} & 1 \\ 6 & \mathbf{0} & 0 \end{bmatrix} \quad \begin{matrix} X_1+X_3 \\ 4 \\ 3 \\ 5 \\ 6 \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} 0 & \mathbf{3} & 4 \\ 1 & \mathbf{6} & 2 \\ 4 & \mathbf{1} & 1 \\ 6 & \mathbf{0} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & \mathbf{4} \\ 1 & 6 & \mathbf{2} \\ 4 & 1 & \mathbf{1} \\ 6 & 0 & \mathbf{0} \end{bmatrix} \quad \begin{matrix} X_1+X_2 \\ 3 \\ 7 \\ 5 \\ 6 \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} 0 & 3 & \mathbf{4} \\ 1 & 6 & \mathbf{0} \\ 4 & 1 & \mathbf{2} \\ 6 & 0 & \mathbf{1} \end{bmatrix}$$

Aggregate risk with minimum variance

Each column is antimonotonic with the sum of the others:

$$\begin{array}{c}\downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix}\end{array} \begin{array}{c} X_2+X_3 \\ 7 \\ 6 \\ 3 \\ 1 \end{array}, \quad \begin{array}{c}\downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix}\end{array} \begin{array}{c} X_1+X_3 \\ 4 \\ 1 \\ 6 \\ 7 \end{array}, \quad \begin{array}{c}\downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix}\end{array} \begin{array}{c} X_1+X_2 \\ 3 \\ 7 \\ 5 \\ 6 \end{array}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{6} & \mathbf{0} \\ \mathbf{4} & \mathbf{1} & \mathbf{2} \\ \mathbf{6} & \mathbf{0} & \mathbf{1} \end{bmatrix} \qquad S_N = \overset{X_1+X_2+X_3}{\begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}}$$

The minimum variance of the sum is equal to 0! (ideal case of a constant sum (*complete mixability*, see Wang and Wang (2011))

Bounds on variance

Analytical Bounds on Standard Deviation

Consider d risks X_i with standard deviation σ_i

$$0 \leqslant std(X_1 + X_2 + ... + X_d) \leqslant \sigma_1 + \sigma_2 + ... + \sigma_d$$

Example with 20 normal N(0,1)

$$0 \leqslant std(X_1 + X_2 + ... + X_{20}) \leqslant 20$$

and in this case, both bounds are sharp and too wide for practical use!

Our idea: Incorporate information on dependence.

Illustration with 2 risks with marginals N(0,1)

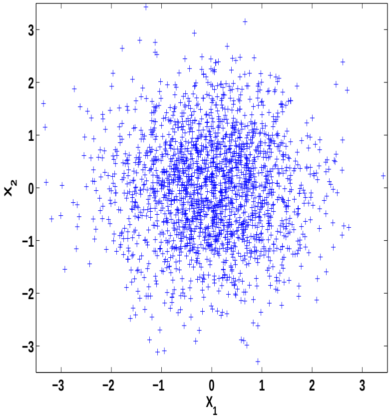
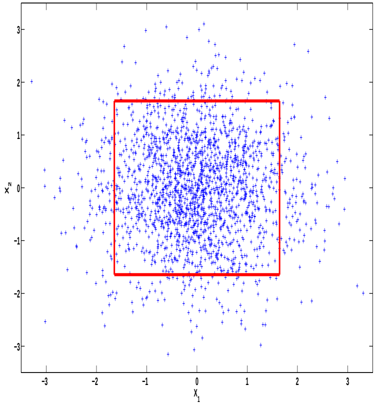


Illustration with 2 risks with marginals N(0,1)



Assumption: Independence on $\mathcal{F} = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$

Our assumptions on the cdf of $(X_1, X_2, ..., X_d)$

$\mathcal{F} \subset \mathbb{R}^d$ ("trusted" or "fixed" area)
$\mathcal{U} = \mathbb{R}^d \backslash \mathcal{F}$ ("untrusted").
We assume that we know:

(i) the marginal distribution F_i of X_i on $\mathbb{R}$ for $i = 1, 2, ..., d$,
(ii) the distribution of $(X_1, X_2, ..., X_d) \mid \{(X_1, X_2, ..., X_d) \in \mathcal{F}\}$.
(iii) $P((X_1, X_2, ..., X_d) \in \mathcal{F})$

- When only marginals are known: $\mathcal{U} = \mathbb{R}^d$ and $\mathcal{F} = \emptyset$.
- **Our Goal: Find bounds on $\rho(S) := \rho(X_1 + ... + X_d)$ when $(X_1, ..., X_d)$ satisfy (i), (ii) and (iii).**

Example:

$N = 8$ observations, $d = 3$ dimensions
and 3 observations trusted ($\ell_f = 3$, $p_f = 3/8$)

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad S_N = \begin{bmatrix} 8 \\ 3 \\ 5 \\ 3 \\ 8 \\ 4 \\ 4 \\ 9 \end{bmatrix}$$

Example: $N = 8$, $d = 3$ with 3 observations trusted ($\ell_f = 3$)

Maximum variance:

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 10 \\ 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

Minimum variance:

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

Example $d = 20$ risks N(0,1)

- $(X_1, ..., X_{20})$ independent N(0,1) on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \qquad p_f = P\left((X_1, ..., X_{20}) \in \mathcal{F}\right)$$

(for some $\beta \leqslant 50\%$) where q_γ: γ-quantile of N(0,1)

- $\beta = 0\%$: no uncertainty (20 independent N(0,1))
- $\beta = 50\%$: full uncertainty

$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$	$\mathcal{U} = \emptyset$ $\beta = 0\%$			$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
$\rho = 0$	4.47			(0 , 20)

Example $d = 20$ risks N(0,1)

- $(X_1, ..., X_{20})$ independent N(0,1) on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \qquad p_f = P\left((X_1, ..., X_{20}) \in \mathcal{F}\right)$$

(for some $\beta \leqslant 50\%$) where q_γ: γ-quantile of N(0,1)

- $\beta = 0\%$: no uncertainty (20 independent N(0,1))
- $\beta = 50\%$: full uncertainty

$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$p_f \approx 98\%$ $\beta = 0.05\%$	$p_f \approx 82\%$ $\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
$\rho = 0$	4.47	(4.4 , 5.65)	(3.89 , 10.6)	(0 , 20)

Model risk on the volatility of a portfolio is reduced a lot by incorporating information on dependence!

Bounds on Value-at-Risk

Part 1 works for all risk measures that satisfy convex order... But not for Value-at-Risk.

- VaR_q is **not** maximized for the comonotonic scenario:

$$S^c = X_1^c + X_2^c + ... + X_d^c$$

  where all X_i^c are *comonotonic.*

- to maximize VaR_q, the idea is to change the comonotonic dependence such that the sum is constant in the tail

Bounds on Value-at-Risk

Part 1 works for all risk measures that satisfy convex order... But not for Value-at-Risk.

- VaR_q is **not** maximized for the comonotonic scenario:

$$S^c = X_1^c + X_2^c + ... + X_d^c$$

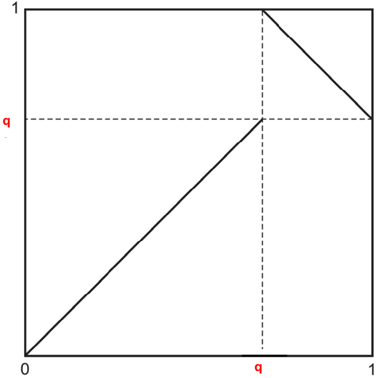
  where all X_i^c are *comonotonic.*

- to maximize VaR_q, the idea is to change the comonotonic dependence such that the sum is constant in the tail

Let us illustrate the problem with two risks:
If X_1 and X_2 are Uniform (0,1) and comonotonic, then

$$VaR_q(S^c) = 2q$$

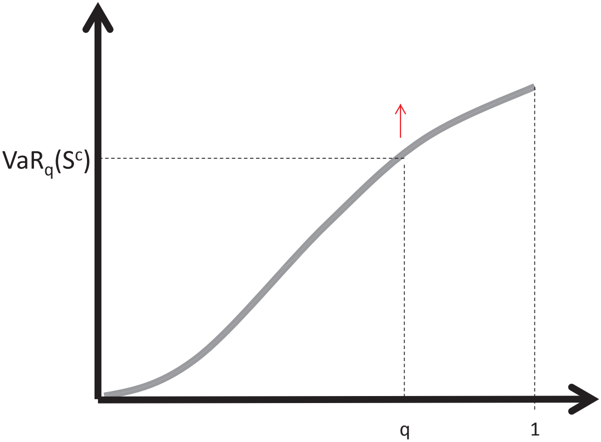
"Riskiest" Dependence Structure maximum VaR at level q in 2 dimensions



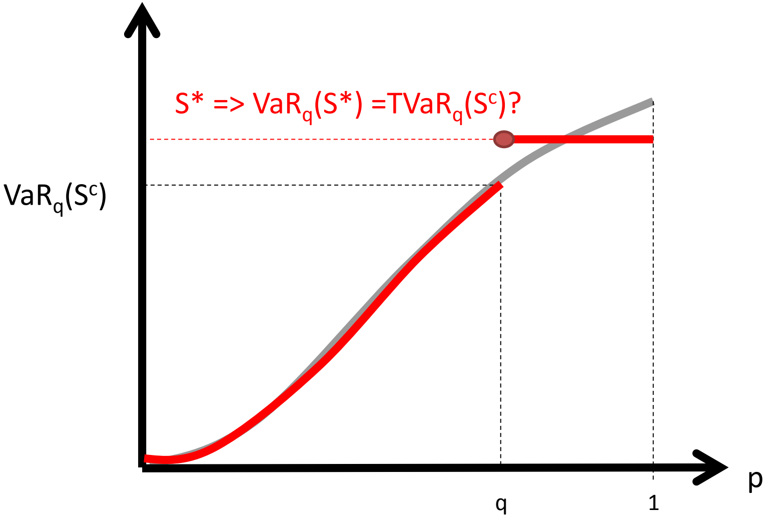
For that dependence structure (*antimonotonic in the tail*)

$$VaR_q(S^*) = 1 + q > VaR_q(S^c) = 2q$$

VaR at level q of the comonotonic sum w.r.t. q



Riskiest Dependence Structure VaR at level q



Numerical Results, 20 risks $N(0,1)$

- VaR of the sum of 20 *independent* N(0,1).

$$\text{VaR}_{95\%} = 12.5 \qquad \text{VaR}_{99.95\%} = 25.1$$

- Bounds on VaR_q for a portfolio of 20 risks N(0,1).

q=95%	(-2.17 , 41.3)
q=99.95%	(-0.035 , 71.1)

- Model risk on dependence is huge!

Our idea: add information on dependence from a fitted model where data is available...

Numerical Results, 20 independent $N(0,1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$

	$\mathcal{U} = \emptyset$ $\beta = 0\%$			$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
q=95%	12.5			(-2.17 , 41.3)
q=99.95%	25.1			(-0.035 , 71.1)

Numerical Results, 20 independent $N(0,1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$

	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$p_f \approx 98\%$ $\beta = 0.05\%$	$p_f \approx 82\%$ $\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
q=95%	12.5	(12.2 , 13.3)	(10.7 , 27.7)	(-2.17 , 41.3)
q=99.95%	25.1	(24.2 , 71.1)	(21.5 , 71.1)	(-0.035 , 71.1)

- **The risk for an underestimation of VaR is increasing in the probability level used to assess the VaR.**
- **For VaR at high probability levels ($q = 99.95\%$), despite all the added information on dependence, the bounds are still wide!**

Conclusions

- Assess model risk with partial information and given marginals
- Results for VaR:
  - Maximum VaR is not the comonotonic scenario.
  - Maximum VaR corresponds to minimum variance in the tail.
  - Bounds on VaR at high confidence level stay wide even if the multivariate dependence is known in 98% of the space!
- Challenges:
  - How to choose the trusted area $\mathcal{F}$ optimally?
  - Re-discretizing using the fitted marginal $\hat{f}_i$ to increase N
  - Amplify the tails of the margins with a probability distortion
- Additional information on dependence can be incorporated
  - expert opinions on the dependence under some scenarios.
  - variance of the sum (work with Rüschendorf and Vanduffel).
  - higher moments (work with Denuit and Vanduffel)

Acknowledgments

- Society of Actuaries Center of Actuarial Excellence Research Grant
- Research project on "*Risk Aggregation and Diversification*" with Steven Vanduffel for the Canadian Institute of Actuaries.
- Project on *"Systemic Risk"* funded by the Global Risk Institute in Financial Services.
- Natural Sciences and Engineering Research Council of Canada

References

- Bernard, C., Vanduffel S. (2014): "A new approach to assessing model risk in high dimensions", *available on SSRN.*
- Bernard, C., M. Denuit, and S. Vanduffel (2014): "Measuring Portfolio Risk under Partial Dependence Information," *Working Paper*.
- Bernard, C., X. Jiang, and R. Wang (2014): "Risk Aggregation with Dependence Uncertainty," *Insurance: Mathematics and Economics*.
- Bernard, C., Y. Liu, N. MacGillivray, and J. Zhang (2013): "Bounds on Capital Requirements For Bivariate Risk with Given Marginals and Partial Information on the Dependence," *Dependence Modelling*.
- Bernard, C., L. Rüschendorf, and S. Vanduffel (2014): "VaR Bounds with a Variance Constraint," *Working Paper*.
- Embrechts, P., G. Puccetti, and L. Rüschendorf (2013): "Model uncertainty and VaR aggregation," *Journal of Banking & Finance*.
- Puccetti, G., and L. Rüschendorf (2012): "Computation of sharp bounds on the distribution of a function of dependent risks," *Journal of Computational and Applied Mathematics*, 236(7), 1833–1840.
- Wang, B., and R. Wang (2011): "The complete mixability and convex minimization problems with monotone marginal densities," *Journal of Multivariate Analysis*, 102(10), 1344–1360.